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Direct and inverse estimates for a new family of linear positive operators

M.K. Gupta [a,*], Vijay Gupta [b,1], Manoj Kumar [c]

[a] *Department of Mathematics, Ch. Charan Singh University, Meerut 250004, India*
[b] *Department of Mathematics and Statistics, Auburn University, Auburn 36849-5108, Alabama, USA*
[c] *Department of Mathematics and Statistics, Gurukula Kangri University, Hardwar 249404, India*

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Abstract

In the present paper we introduce a new family of linear positive operators and study some direct and inverse results in simultaneous approximation.

Keywords: Simultaneous approximation; Beta operators; Linear positive operators; Steklov mean; Modulus of continuity

1. Introduction

Let $C_\gamma[0,\infty)$, where $\gamma > 0$, be the class of all continuous functions defined on $[0,\infty)$ satisfying the growth condition $|f(t)| \leqslant Ct^\gamma$ for some positive constant C. For $f \in C_\gamma[0,\infty)$ we define a new family $\{B_n f\}$ of linear positive operators of the form

$$B_n(f,x) \equiv B_n\big(f(t),x\big) = \frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty b_{n,v}(t)f(t)\,dt, \quad x \in [0,\infty), \tag{1.1}$$

* Corresponding author.
E-mail addresses: mkgupta2002@hotmail.com (M.K. Gupta), vijaygupta2001@hotmail.com (V. Gupta), kangrimk_77@yahoo.com (M. Kumar).
[1] Current address: School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi 110075, India.

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where

$$b_{n,v}(t)=\frac{1}{B(v,n+1)}t^{v-1}(1+t)^{-n-v-1}$$

with $B(v,n+1)=(\nu-1)!n!/(n+\nu)!$ the Beta function. The norm $\|.\|_\gamma$ on $C_\gamma[0,\infty)$ is defined as $\|f\|_\gamma=\sup_{0<t<\infty}|f(t)|t^{-\gamma}$. Let $W_n(x,t)=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)b_{n,v}(t)$, then operators (1.1) may be written in the following form:

$$B_n(f,x)=\int_0^\infty W_n(x,t)f(t)\,dt.$$

It is easily checked that the operators B_n defined above are linear positive operators and that $B_n(1,x)=1$. It turns out that the order of approximation by these operators is at best $O(n^{-1})$ as $n\to\infty$, howsoever smooth the function f may be. For some other works in this direction, we refer the readers to [4,7]. Throughout this paper, we denote by $C[a,b]$ the space of all continuous functions on the interval $[a,b]$, the norm $\|.\|_{C[a,b]}$ denotes the sup norm on the space $C[a,b]$.

For $f\in C[a,b]$ and a positive integer $k\geqslant 1$, the kth order modulus of continuity is defined as

$$\omega_k(f,\delta;a,b)=\sup\bigl\{\bigl|\Delta_h^k f(x)\bigr|:\ |h|\leqslant\delta \text{ and } x,x+kh\in[a,b]\bigr\},$$

where $\Delta_h^k f(x)$ is the kth forward difference with step length h.

A function f is said to belong to the generalized Zygmund class $\mathrm{Liz}(\alpha,k;a,b)$ if for $\delta>0$ there exists a constant C such that

$$\omega_{2k}(f,\delta;a,b)\leqslant C\delta^{\alpha k}.$$

In particular for $k=1$, we simply write $\mathrm{Lip}^*(\alpha,a,b)$ instead of $\mathrm{Liz}(\alpha,1;a,b)$. By C_0 we mean the class of continuous functions defined on $(0,\infty)$ having a compact support and C_0^r the subclass of C_0, consisting of r-times continuously differentiable functions with $\mathrm{supp}[a',b']\subset(a,b)$ and $[a,b]\subset(0,\infty)$. Also let

$$G^{(r)}=\bigl\{g\in C_0^{r+2}:\ \mathrm{supp}\,g\subset[a',b']\bigr\}.$$

For $f\in C_0^r$ with $\mathrm{supp}\,f\subset[a',b']$, the Peetre's K-functionals are defined as

$$\begin{aligned}K_r(\xi,f;a,b)=\inf_{g\in G^{(r)}}\bigl\{\bigl\|f^{(r)}-g^{(r)}\bigr\|_{C[a',b']}+\xi\bigl(\bigl\|g^{(r)}\bigr\|_{C[a',b']}\\ +\bigl\|g^{(r+2)}\bigr\|_{C[a',b']}\bigr)\bigr\},\quad 0<\xi\leqslant 1.\end{aligned}$$

For $0<\alpha<2$ and $f\in C_0^r$ with $\mathrm{supp}\,f\subset[a',b']$, we say that $f\in C_0^r(\alpha,k+1;a',b')$ if

$$\|f\|_{\alpha,r}\equiv\sup_{0<\xi\leqslant 1}\xi^{-\alpha/2}K_r(\xi,f)<\infty.$$

The main aim of the present paper is to study pointwise convergence, Voronovskaja-type asymptotic formula, an error estimate and an inverse estimate, in simultaneous approximation by the operators (1.1).

2. Preliminary results

In the sequel, we shall require the following results:

Lemma 2.1. [5] *Let $m \in N^0$, the set of non-negative integers. If the mth order moment for the operators* (1.1) *be defined as*

$$U_{n,m}(x) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \left(\frac{v-1}{n+2} - x \right)^m,$$

then we have $U_{n,0}(x) = 1$, $U_{n,1}(x) = 0$ and there holds the recurrence relation

$$(n+2) U_{n,m+1}(x) = x(1+x) \left[U'_{n,m}(x) + m U_{n,m-1}(x) \right].$$

Consequently, for each $x \in [0, \infty)$, it is easily verified from the above recurrence relation that $U_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

Lemma 2.2. *For $m \in N^0$ and $n > m$, let the function $\mu_{n,m}(x)$ be defined as*

$$\mu_{n,m}(x) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^m \, dt.$$

Then

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{2x+1}{n},$$

and there holds the recurrence relation

$$(n-m) \mu_{n,m+1}(x) = x(1+x) \left[\mu'_{n,m}(x) + 2m \mu_{n,m-1}(x) \right] + (m+1)(1+2x) \mu_{n,m}(x).$$

Consequently, for each $x \in [0, \infty)$, we have $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$.

Proof. Since $\mu_{n,m}(x) = B_n((t-x)^m, x)$, therefore by linearity of the operators, we have

$$\mu_{n,0}(x) = B_n\big((t-x)^0, x\big) = B_n(1, x) = 1$$

and

$$\mu_{n,1}(x) = B_n\big((t-x), x\big) = B_n(t, x) - x B_n(1, x) = \frac{1+2x}{n}.$$

Next, making use of the identity $x(1+x) b'_{n,v}(x) = [(v-1) - (n+2)x] b_{n,v}(x)$, we have

$$x(1+x) \mu'_{n,m}(x) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} x(1+x) b'_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^m \, dt$$
$$- m \frac{1}{(n+1)} \sum_{v=1}^{\infty} x(1+x) b_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^{m-1} \, dt.$$

Thus

$$x(1+x) \left[\mu'_{n,m}(x) + m \mu_{n,m-1}(x) \right]$$
$$= \frac{1}{(n+1)} \sum_{v=1}^{\infty} \left[(v-1) - (n+2)x \right] b_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^m \, dt$$

$$
\begin{aligned}
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} \big[(v-1)-(n+2)t+(n+2)(t-x)\big] b_{n,v}(t)(t-x)^m\,dt \\
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} t(1+t) b'_{n,v}(t)(t-x)^m\,dt + (n+2)\mu_{n,m+1}(x) \\
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} \big[(t-x)^2+(1+2x)(t-x)+x(1+x)\big] b'_{n,v}(t)(t-x)^m\,dt \\
&\quad + (n+2)\mu_{n,m+1}(x) \\
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b'_{n,v}(t)(t-x)^{m+2}\,dt \\
&\quad + \frac{(1+2x)}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b'_{n,v}(t)(t-x)^{m+1}\,dt \\
&\quad + \frac{x(1+x)}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b'_{n,v}(t)(t-x)^{m}\,dt + (n+2)\mu_{n,m+1}(x) \\
&= -(m+2)\mu_{n,m+1}(x) - (m+1)(1+2x)\mu_{n,m}(x) \\
&\quad - mx(1+x)\mu_{n,m-1}(x) + (n+2)\mu_{n,m+1}(x).
\end{aligned}
$$

Therefore

$$
(n-m)\mu_{n,m+1}(x) = (m+1)(1+2x)\mu_{n,m}(x) + x(1+x)\big[\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)\big].
$$

This completes the proof of the recurrence relation. □

Corollary 2.3. *For $n > i$ and each $x \in (0,\infty)$, it can be easily verified from Lemma* 2.2 *and by mathematical induction that*

$$
B_n\big(t^i, x\big) = \frac{(n+i+1)!(n-i)!}{n!(n+1)!} x^i + i^2 \frac{(n+i)!(n-i)!}{n!(n+1)!} x^{i-1} + i(i-1)O\big(n^{-2}\big).
$$

Making use of Taylor's expansion, Schwarz inequality for summation and integration and Lemma 2.2, we can easily prove the following lemma.

Lemma 2.4. *Let δ and γ be any two positive real numbers. Then for every $n > \gamma$ and each $x \in (0,\infty)$ there exists a constant $C(k,x)$ independent of n and depending on k and x such that*

$$
\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_{|t-x|>\delta} b_{n,v}(t) t^{\gamma}\,dt \leqslant C(k,x) n^{-k}, \quad k = 1,2,3,\ldots.
$$

Lemma 2.5. [5] *There exist the polynomials $Q_{i,j,r}(x)$ independent of n and v such that*

$$\left[x(1+x)\right]^r D^r b_{n,v}(x) = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i\left[(v-1)-(n+2)x\right]^j Q_{i,j,r}(x) b_{n,v}(x),$$

where $D \equiv \frac{d}{dx}$.

Lemma 2.6. *Let the function f be r-times differentiable on $[0,\infty)$ such that $f^{(r-1)}$ is absolutely continuous with $f^{(r-1)}(t) = O(t^\gamma)$ for some $\gamma > 0$ as $t \to \infty$. Then for $r = 1, 2, 3, \dots$ and $n > \gamma + r$,*

$$B_n^{(r)}(f,x) = \frac{(n+r)!(n-r)!}{n!(n+1)!} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) f^{(r)}(t)\,dt.$$

Proof. We have by Leibniz theorem

$$\begin{aligned}
B_n^{(r)}(f,x) &= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}^{(r)}(x) \int_0^\infty b_{n,v}(t) f(t)\,dt \\
&= \frac{1}{(n+1)} \sum_{i=0}^{r} \sum_{v=i+1}^{\infty} \frac{(n+v)!}{(v-1)!n!} \binom{r}{i} \left(D^i x^{v-1}\right)\left(D^{r-i}(1+x)^{-(n+v+1)}\right) \\
&\quad \times \int_0^\infty b_{n,v}(t) f(t)\,dt \\
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \frac{(n+r)!}{n!} \frac{(n+v+r)!}{(v-1)!(n+r)!} \frac{x^{v-1}}{(1+x)^{n+r+v+1}} \\
&\quad \times \int_0^\infty b_{n,v+i}(t) f(t)\,dt \\
&= \frac{(n+r)!}{n!(n+1)} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty (-1)^r \left\{ \sum_{i=0}^{r} (-1)^i \binom{r}{i} b_{n,v+i}(t) \right\} f(t)\,dt. \qquad (2.1)
\end{aligned}$$

Again by Leibniz theorem, we have

$$\begin{aligned}
b_{n-r,v+r}^{(r)}(t) &= \sum_{i=0}^{r} \frac{(n+v)!}{(n-r)!(v+r-1)!} \binom{r}{i} \left(D^{r-i} t^{v+r-1}\right)\left(D^i (1+t)^{-(n+v+1)}\right) \\
&= \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{n!}{(n-r)!} \frac{(n+v+i)!}{(v+i-1)!n!} \frac{t^{v+i-1}}{(1+t)^{n+v+i+1}} \\
&= \frac{n!}{(n-r)!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} b_{n,v+i}(t). \qquad (2.2)
\end{aligned}$$

Thus from (2.1) and (2.2), we get

$$B_n^{(r)}(f,x)=\frac{(n+r)!(n-r)!}{n!(n+1)!}\sum_{v=1}^{\infty}b_{n+r,v}(x)\int_0^{\infty}(-1)^r b_{n-r,v+r}^{(r)}(t)f(t)\,dt.$$

Further integrating by parts r times, we get the required result. □

Lemma 2.7. [2,6] *For sufficiently small $\delta>0$ and $0<a<a_1<b_1<b<\infty$, the mth order Steklov mean $f_{m,\delta}(t)$ corresponding to $f\in C_\gamma[0,\infty)$ is defined by*

$$f_{m,\delta}(t)=\delta^{-m}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\left[f(t)+(-1)^{m-1}\Delta_\eta^m f(t)\right]dt_1\,dt_2\cdots dt_m,$$

where $\eta=\frac{1}{m}\sum_{i=1}^m t_i$, $t\in[a,b]$ and $\Delta_\eta^m f(t)$ is the mth forward difference with step length η.

The function $f_{m,\delta}(t)$ has the following properties:

(i) $f_{m,\delta}$ has continuous derivatives up to order m on $[a,b]$,
(ii) $\|f_{m,\delta}^{(r)}\|_{C[a_1,b_1]}\leqslant C_1\delta^{-r}\omega_r(f,\delta;a_1,b_1)$, $r=1,2,3,\ldots,m$,
(iii) $\|f-f_{m,\delta}\|_{C[a_1,b_1]}\leqslant C_2\omega_m(f,\delta;a,b)$,
(iv) $\|f_{m,\delta}\|_{C[a_1,b_1]}\leqslant C_3\|f\|_\gamma$,

where C_1,C_2,C_3 are certain unrelated constants depending on m and independent of f and δ. Further the constants C_i are not necessarily the same in different places.

3. Direct results

In this section we study the rate of pointwise convergence, an asymptotic formula and error estimation in terms of higher order modulus of continuity in simultaneous approximation for the operators (1.1).

Theorem 3.1. *Let $f\in C_\gamma[0,\infty)$, $\gamma>0$, and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then*

$$B_n^{(r)}(f,x)=f^{(r)}(x)\quad \text{as } n\to\infty.$$

Proof. By Taylor's finite expansion of f, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,\quad \text{where } \varepsilon(t,x)\to 0 \text{ as } t\to x.$$

Thus applying the above expansion, we have

$$\begin{aligned}B_n^{(r)}(f,x)&=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^{\infty}W_n^{(r)}(t,x)(t-x)^i\,dt+\int_0^{\infty}W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt\\&=I_1+I_2,\quad \text{say.}\end{aligned}$$

Using binomial expansion of $(t-x)^i$, Lemma 2.2 and Corollary 2.3, we have

$$\begin{aligned}
I_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{\mathrm{n}}^{(r)}(t,x) \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} t^j \, dt \\
&= \frac{f^{(r)}(x)}{r!} \frac{\partial^r}{\partial x^r} \int_0^\infty W_n(t,x) t^r \, dt \\
&= \frac{f^{(r)}(x)}{r!} \frac{\partial^r}{\partial x^r} \left\{ \frac{(n+r+1)!(n-r)!}{n!(n+1)!} x^r + \text{terms containing powers of } x \text{ less than } r \right\} \\
&= \frac{f^{(r)}(x)}{r!} \left\{ \frac{(n+r+1)!(n-r)!}{n!(n+1)!} r! + 0 \right\} = f^{(r)}(x) \quad \text{as } n \to \infty.
\end{aligned}$$

Next using Lemma 2.5, we get

$$\begin{aligned}
I_2 &= \int_0^\infty W_n^{(r)}(t,x) \varepsilon(t,x)(t-x)^r \, dt \\
&= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}^{(r)}(x) \int_0^\infty b_{n,v}(t,)\varepsilon(t,x)(t-x)^r \, dt \\
&= \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} \frac{Q_{i,j,r}(x)}{[x(1+x)]^r} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} \big[(v-1)-(n+2)x\big]^j b_{n,v}(x) \\
&\quad \times \int_0^\infty b_{n,v}(t)\varepsilon(t,x)(t-x)^r \, dt.
\end{aligned}$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$ therefore, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. Thus for some constants C_1 and C_2, we can write

$$\begin{aligned}
|I_2| &\leqslant C_1 \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \big|(v-1)-(n+2)x\big|^j \\
&\quad \times \left[\varepsilon \int_{|t-x|<\delta} b_{n,v}(t)|t-x|^r \, dt + \int_{|t-x| \geqslant \delta} b_{n,v}(t) C_2 t^\gamma \, dt \right] \\
&= I_3 + I_4, \quad \text{say,}
\end{aligned}$$

where

$$C_1 = \sup_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} \frac{|Q_{i,j,r}(x)|}{[x(1+x)]^r}$$

and C_2 is independent of t. Now using Schwarz inequality for integration and summation, Lemmas 2.1 and 2.2, we get

$$\begin{aligned}
I_3 &\leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \sum_{v=1}^{\infty} \frac{b_{n,v}(x)}{(n+1)} \big|(v-1)-(n+2)x\big|^j \\
&\quad \times \left(\int_0^{\infty} b_{n,v}(t)\,dt\right)^{1/2} \left(\int_0^{\infty} b_{n,v}(t)(t-x)^{2r}\,dt\right)^{1/2} \\
&\leqslant \varepsilon C_1 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \left(\sum_{v=1}^{\infty} \frac{b_{n.v}(x)}{(n+1)} \big[(v-1)-(n+2)x\big]^{2j}\right)^{1/2} \\
&\quad \times \left(\sum_{v=1}^{\infty} \frac{b_{n,v}(x)}{(n+1)} \int_0^{\infty} b_{n,v}(t)(t-x)^{2r}\,dt\right)^{1/2} \\
&\leqslant \varepsilon C \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i\, O\big(n^{j/2}\big) O\big(n^{-r/2}\big) = \varepsilon O(1).
\end{aligned}$$

Again using Schwarz inequality, Lemmas 2.1 and 2.4, we get

$$\begin{aligned}
I_4 &\leqslant C_1 C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \big|(v-1)-(n+2)x\big|^j \int_{|t-x|\geqslant\delta} b_{n,v}(t) t^{\gamma}\,dt \\
&\leqslant C_1 C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \big|(v-1)-(n+2)x\big|^j \\
&\quad \times \left(\int_{|t-x|\geqslant\delta} b_{n,v}(t)\,dt\right)^{1/2} \left(\int_{|t-x|\geqslant\delta} b_{n,v}(t) t^{\gamma}\,dt\right)^{1/2} \\
&\leqslant C_1 C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \left(\sum_{v=1}^{\infty} \frac{b_{n,v}(x)}{(n+1)} \big[(v-1)-(n+2)x\big]^{2j}\right)^{1/2} \\
&\quad \times \left(\sum_{v=1}^{\infty} \frac{b_{n,v}(x)}{(n+1)} \int_0^{\infty} b_{n,v}(t) t^{2\gamma}\,dt\right)^{1/2} \\
&= C_1 C_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i\, O\big(n^{j/2}\big) O\big(n^{-k/2}\big) \quad \text{for any } k>0.
\end{aligned}$$

Choosing $k>r$ we get $I_4 = o(1)$. Also due to arbitrariness of $\varepsilon>0$ it follows that $I_3 = o(1)$. Hence $I_2 = o(1)$. Finally collecting the estimates of I_1 and I_2, we get the required result. □

Theorem 3.2. *Let $f \in C_{\gamma}[0,\infty)$, $\gamma>0$. If $f^{(r+2)}$ exists at a point $x\in(0,\infty)$, then*

$$\begin{aligned}
&\lim_{n\to\infty} n\big[B_n^{(r)}(f,x) - f^{(r)}(x)\big] \\
&\quad = r(r+1) f^{(r)}(x) + (2x+1)(r+1) f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).
\end{aligned}$$

Proof. Using Taylor's finite expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and $\varepsilon(t,x)=O((t-x)^{\alpha})$ as $t\to\infty$ for some $\alpha>0$. Using Lemma 2.2, we get

$$\begin{aligned}
n\big[B_n^{(r)}(f,x)-f^{(r)}(x)\big]&=n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right]\\
&\quad+\left[n\int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}\,dt\right]\\
&=E_1+E_2,\quad\text{say.}
\end{aligned}$$

Now using Corollary 2.3, we get

$$\begin{aligned}
E_1&=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}t^j\,dt-nf^{(r)}(x)\\
&=n\frac{f^{(r)}(x)}{r!}\big[B_n^{(r)}\big(t^r,x\big)-(r!)\big]\\
&\quad+n\frac{f^{(r+1)}(x)}{(r+1)!}\big[(r+1)(-x)B_n^{(r)}\big(t^r,x\big)+B_n^{(r)}\big(t^{r+1},x\big)\big]\\
&\quad+n\frac{f^{(r+2)}(x)}{(r+2)!}\bigg[\frac{(r+2)(r+1)}{2}x^2B_n^{(r)}\big(t^r,x\big)\\
&\quad+(r+2)(-x)B_n^{(r)}\big(t^{r+1},x\big)+B_n^{(r)}\big(t^{r+2},x\big)\bigg]\\
&=nf^{(\mathrm{r})}(x)\left[\frac{(n+r+1)!(n-r)!}{n!(n+1)!}-1\right]\\
&\quad+n\frac{f^{(r+1)}(x)}{(r+1)!}\bigg[-x(r+1)\frac{(n+r+1)!(n-r)!}{n!(n+1)!}r!\\
&\quad+\frac{(n+r+2)!(n-r-1)!}{n!(n+1)!}(r+1)!x+(r+1)^2\frac{(n+r+1)!(n-r-1)!}{n!(n+1)!}r!\bigg]\\
&\quad+n\frac{f^{(r+2)}(x)}{(r+2)!}\bigg[\frac{(r+2)(r+1)x^2}{2}\frac{(n+r+1)!(n-r)!}{n!(n+1)!}r!\\
&\quad-(r+2)x\frac{(n+r+2)!(n-r-1)!}{n!(n+1)!}(r+1)!x\\
&\quad-(r+2)(r+1)^2x\frac{(n+r+1)!(n-r-1)!}{n!(n+1)!}r!\\
&\quad+\frac{(n+r+3)!(n-r-2)!}{n!(n+1)!}\frac{(r+2)!}{2}x^2
\end{aligned}$$

$$+ (r+2)^2 \frac{(n+r+2)!(n-r-2)!}{n!(n+1)!}(r+1)!x\biggr] + O\bigl(n^{-2}\bigr)$$
$$= r(r+1)f^{(r)}(x) + (2x+1)(r+1)f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x) \quad \text{as } n \to \infty.$$

In order to complete the proof of the theorem it is sufficient to show that $E_2 \to 0$ as $n \to \infty$, which can be easily proved along the lines of the proof of Theorem 3.1 using Lemmas 2.1, 2.2 and 2.5. □

Theorem 3.3. *Let $f \in C_\gamma[0,\infty)$, $\gamma > 0$, and $0 < a < a_1 < b_1 < b < \infty$. Then for all n sufficiently large, we have*

$$\bigl\| B_n^{(r)}(f,*) - f^{(r)} \bigr\|_{C[a_1,b_1]} \leqslant A_1 \omega_2\bigl(f^{(r)}, n^{-1/2}; a, b\bigr) + A_2 n^{-1} \|f\|_\gamma,$$

where $A_1 = A_1(r)$ and $A_2 = A_2(r, f)$.

Proof. Using linearity property, we have

$$\begin{aligned}
&\bigl\| B_n^{(r)}(f,*) - f^{(r)} \bigr\|_{C[a_1,b_1]} \\
&\quad \leqslant \bigl\| B_n^{(r)}\bigl((f - f_{2,\delta}),*\bigr) \bigr\|_{C[a_1,b_1]} + \bigl\| B_n^{(r)}(f_{2,\delta},*) - f_{2,\delta}^{(r)} \bigr\|_{C[a_1,b_1]} + \bigl\| f^{(r)} - f_{2,\delta}^{(r)} \bigr\|_{C[a_1,b_1]} \\
&\quad = J_1 + J_2 + J_3, \quad \text{say.}
\end{aligned}$$

Using property (iii) of Lemma 2.7 and Theorem 3.2, respectively, we get

$$J_3 \leqslant C_1 \omega_2\bigl(f^{(r)}, \delta; a, b\bigr) \quad \text{and} \quad J_2 \leqslant C_2 n^{-1} \sum_{j=r}^{r+2} \bigl\| f_{2,\delta}^{(j)} \bigr\|_{C[a,b]}.$$

Next using the interpolation property due to Goldberg and Meir [3] for each $j = r, r+1, r+2$, we have

$$\bigl\| f_{2,\delta}^{(j)} \bigr\|_{C[a,b]} \leqslant C_3 \bigl(\|f_{2,\delta}\|_{C[a,b]} + \bigl\| f_{2,\delta}^{(r+2)} \bigr\|_{C[a,b]} \bigr).$$

Using properties (ii) and (iv) of Lemma 2.7, we get

$$J_2 \leqslant C_4 n^{-1} \bigl(\|f\|_\gamma + \delta^{-2} \omega_2\bigl(f^{(r)}, \delta; a, b\bigr)\bigr).$$

To estimate J_1, we denote the characteristic function of the interval $[a', b']$ by $\psi(t)$, where $0 < a < a' < a_1 < b_1 < b' < b < \infty$. Then we have

$$\begin{aligned}
J_1 &\leqslant \bigl\| B_n^{(r)}\bigl(\psi(t)\bigl[f(t) - f_{2,\delta}(t)\bigr],*\bigr) \bigr\|_{C[a_1,b_1]} \\
&\quad + \bigl\| B_n^{(r)}\bigl(\bigl[1 - \psi(t)\bigr]\bigl[f(t) - f_{2,\delta}(t)\bigr],*\bigr) \bigr\|_{C[a_1,b_1]} \\
&= J_4 + J_5, \quad \text{say.}
\end{aligned}$$

Using Lemma 2.6, we have

$$\begin{aligned}
&B_n^{(r)}\bigl(\psi(t)\bigl[f(t) - f_{2,\delta}(t)\bigr], x\bigr) \\
&\quad = \frac{(n+r)!(n-r)!}{n!(n+1)!} \sum_{v=1}^{\infty} b_{n+r,v}(x) \int_0^\infty b_{n-r,v+r}(t) \frac{d^r}{dt^r}\bigl(\psi(t)\bigl[f(t) - f_{2,\delta}(t)\bigr]\bigr)\, dt
\end{aligned}$$

Hence, using property (iii) of Lemma 2.7, we get

$$\begin{aligned}J_4 &= \left\| B_n^{(r)}\left(\psi(t)\left[f(t) - f_{2,\delta}(t)\right], *\right)\right\|_{C[a_1,b_1]}\\ &\leqslant C_5 \left\| f^{(r)} - f_{2,\delta}^{(r)}\right\|_{C[a',b']} \leqslant C_6\omega_2\left(f^{(r)}, \delta; a, b\right).\end{aligned}$$

Now for $x \in [a_1, b_1]$ and $t \in [0, \infty)\backslash[a', b']$, if we choose $\delta_1 > 0$ such that $|t - x| \geqslant \delta_1$ then by Lemmas 2.1, 2.2, 2.5 and Schwarz inequality, we have

$$\begin{aligned}J &= \left|B_n^{(r)}\left(\left[1 - \psi(t)\right]\left[f(t) - f_{2,\delta}(t)\right], x\right)\right|\\
&\leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \frac{|Q_{i,j,r}(x)|}{[x(1+x)]^r} \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x)\left|(v-1) - (n+2)x\right|^j\\
&\quad \times \int_0^{\infty} b_{n,v}(t)\left[1 - \psi(t)\right]\left|f(t) - f_{2,\delta}(t)w\right| dt\\
&\leqslant C_7\|f\|_\gamma \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x)\left|(v-1) - (n+2)x\right|^j \int_{|t-x|\geqslant\delta_1} b_{n,v}(t)\, dt\\
&\leqslant C_7\|f\|_\gamma \delta_1^{-2k} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \frac{(n+2)^i}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x)\left|(v-1) - (n+2)x\right|^j\\
&\quad \times \int_{|t-x|\geqslant\delta_1} b_{n,v}(t)(t-x)^{2k}\, dt\\
&\leqslant C_7\delta_1^{-2k}\|f\|_\gamma \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \sum_{v=1}^{\infty} \frac{b_{n,v}(x)}{(n+1)}\left|(v-1) - (n+2)x\right|^j \left(\int_0^{\infty} b_{n,v}(t)\, dt\right)^{1/2}\\
&\quad \times \left(\int_0^{\infty} b_{n,v}(t)(t-x)^{4k}\, dt\right)^{1/2}\\
&\leqslant C_7\|f\|_\gamma \delta_1^{-2k} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i \left(\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x)\left[(v-1) - (n+2)x\right]^{2j}\right)^{1/2}\\
&\quad \times \left(\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^{4k}\, dt\right)^{1/2}\\
&= C_7\|f\|_\gamma \delta_1^{-2k} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^i O\left(n^{j/2}\right) O\left(n^{-k}\right) = C_8 n^{-q}\|f\|_\gamma, \quad \text{where } q = k - r/2.\end{aligned}$$

Now choosing $q \geqslant 1$, we get

$$J_5 = \left\| B_n^{(r)}\left(\left[1-\psi(t)\right]\left[f(t) - f_{2,\delta}(t)\right], *\right)\right\|_{C[a_1,b_1]} \leqslant C_8 n^{-1} \|f\|_\gamma.$$

Finally, choosing $\delta = n^{-1/2}$ and collecting the estimates of J_1, J_2, J_3, J_4 and J_5, we get the required result. □

4. Inverse result

To prove inverse theorem, we shall require the following two auxiliary results:

Lemma 4.1. *Let* $0 < a < a' < a'' < b'' < b' < b < \infty$ *and* $f^{(r)} \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$. *If* $\|B_n^{(r)}(f,*) - f^{(r)}\|_{C[a,b]} = O(n^{-\alpha/2})$, *then*

$$K_r(\xi, f) = C_1\left\{n^{-\alpha/2} + n\xi K_r\left(n^{-1}, f\right)\right\}. \tag{4.1}$$

Consequently, $K_r(\xi, f) \leqslant C_2\xi^{\alpha/2}$, *i.e.,* $f \in C_0^r(\alpha, 1; a', b')$, *where* C_1 *and* C_2 *are some positive constants.*

Proof. As usual, to prove (4.1) it is sufficient to show that

$$K_r(\xi, f) \leqslant C_1\left\{n^{-\alpha/2} + n\xi K_r\left(n^{-1}, f\right)\right\} \quad \text{for all } n \text{ sufficiently large.}$$

Since $\operatorname{supp} f \subset [a'', b'']$, therefore by Theorem 3.2 there exists a function $h^{(i)} \in G^{(r)}$ such that for $i = r$ and $r+2$,

$$\left\| B_n^{(i)}(f,*) - h^{(i)}\right\|_{C[a,b]} \leqslant C_3 n^{-1},$$

which implies that

$$\begin{aligned} K_r(\xi, f) \leqslant{} & 3C_3 n^{-1} + \left\| B_n^{(r)}(f,*) - f^{(r)}\right\|_{C[a',b']} \\ & + \xi\left(\left\| B_n^{(r)}(f,*)\right\|_{C[a',b']} + \left\| B_n^{(r+2)}(f,*)\right\|_{C[a',b']}\right). \end{aligned}$$

Thus it is sufficient to show that there exists a constant C_4 such that for each $g \in G^{(r)}$,

$$\left\| B_n^{(r+2)}(f,*)\right\|_{C[a',b']} \leqslant C_4 n\left(\left\| f^{(r)} - g^{(r)}\right\|_{C[a',b']} + n^{-1}\left\| g^{(r+2)}\right\|_{C[a',b']}\right). \tag{4.2}$$

In fact, by linearity property, we have

$$\left\| B_n^{(r+2)}(f,*)\right\|_{C[a',b']} \leqslant \left\| B_n^{(r+2)}(f-g,*)\right\|_{C[a',b']} + \left\| B_n^{(r+2)}(g,*)\right\|_{C[a',b']}. \tag{4.3}$$

By Lemma 2.5, we have

$$\begin{aligned} & \int_0^\infty \left| \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t)\right| dt \\ & \quad \leqslant \sum_{\substack{2i+j \leqslant r+2 \\ i,j \geqslant 0}} \sum_{v=1}^\infty \frac{(n+2)^i}{(n+1)} \frac{|(v-1) - (n+2)x|^j}{[x(1+x)]^{r+2}} \\ & \qquad \times \left|Q_{i,j,r+2}(x)\right| b_{n,v}(x) \int_0^\infty b_{n,v}(t)\, dt. \end{aligned} \tag{4.4}$$

On the other hand, by Lemma 2.2, we get

$$\int_0^\infty \frac{\partial^r}{\partial x^r} W_n(x,t)(t-x)^i\,dt = 0 \quad \text{for } r > i. \tag{4.5}$$

Using Schwarz inequality and Lemma 2.1, we get

$$\left\| B_n^{(r+2)}(f-g,*)\right\|_{C[a',b']} \leqslant C_5 n \left\| f^{(r)} - g^{(r)}\right\|_{C[a',b']}, \tag{4.6}$$

where C_5 is independent of f and g.

By Taylor's finite expansion of $g(t)$, we have

$$g(t) = \sum_{i=0}^{r+1} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{g^{(r+2)}(\zeta)}{(r+2)!}(t-x)^{r+2}, \tag{4.7}$$

where ζ lies between t and x.

Using (4.7) and (4.5), we get

$$\left\| B_n^{(r+2)}(g,*)\right\|_{C[a',b']} \leqslant \frac{\|g^{(r+2)}\|_{C[a',b']}}{(r+2)!} \left\| \int_0^\infty \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t)(t-x)^{r+2}\,dt \right\|_{C[a',b']}. \tag{4.8}$$

Again using (4.4), Lemmas 2.1, 2.2 and Schwarz inequality for integration and then summation, we get

$$\begin{aligned}
I &= \int_0^\infty \left| \frac{\partial^{r+2}}{\partial x^{r+2}} W_n(x,t)\right| (t-x)^{r+2}\,dt \\
&\leqslant \sum_{\substack{2i+j\leqslant r+2\\ i,j\geqslant 0}} \sum_{v=1}^\infty \frac{(n+2)^i}{(n+1)} \frac{|(v-1)-(n+2)x|^j}{[x(1+x)]^{r+2}} \left|Q_{i,j,r+2}(x)\right| b_{n,v}(x) \\
&\quad \times \int_0^\infty b_{n,v}(t)(t-x)^{r+2}\,dt \\
&\leqslant \sum_{\substack{2i+j\leqslant r+2\\ i,j\geqslant 0}} \frac{(n+2)^i |Q_{i,j,r+2}(x)|}{(n+1)[x(1+x)]^{r+2}} \sum_{v=1}^\infty \left|(v-1)-(n+2)x\right|^j b_{n,v}(x) \\
&\quad \times \left(\int_0^\infty b_{n,v}(t)\,dt \right)^{1/2} \left(\int_0^\infty b_{n,v}(t)(t-x)^{2(r+2)}\,dt \right)^{1/2} \\
&\leqslant \sum_{\substack{2i+j\leqslant r+2\\ i,j\geqslant 0}} \frac{(n+2)^i |Q_{i,j,r+2}(x)|}{[x(1+x)]^{r+2}} \left(\sum_{v=1}^\infty \frac{[(v-1)-(n+2)x]^{2j}}{(n+1)} b_{n,v}(x) \right)^{1/2} \\
&\quad \times \left(\sum_{v=1}^\infty \frac{b_{n,v}(x)}{(n+1)} \int_0^\infty b_{n,v}(t)(t-x)^{2(r+2)}\,dt \right)^{1/2}
\end{aligned}$$

$$= C_6 \sum_{\substack{2i+j\leqslant r+2\\ i,j\geqslant 0}} (n+2)^i O\big(n^{j/2}\big) O\big(n^{-(r+2)/2}\big) = C_7 O(1).$$

Hence, we get

$$\big\| B_n^{(r+2)}(g,*)\big\|_{C[a',b']} \leqslant C_8 \big\| g^{(r+2)}\big\|_{C[a',b']}. \tag{4.9}$$

Finally, combining the estimates of (4.3)–(4.9) we get (4.2). This completes the proof of (4.1). The other consequence follows from [1]. □

Lemma 4.2. *Let* $0 < a < a' < a'' < b'' < b' < b < \infty$ *and* $f^{(r)} \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$. *If* $f \in C_0^r(\alpha, 1; a', b')$ *then* $f^{(r)} \in \mathrm{Lip}^*(\alpha, a', b')$.

Proof. Let $|h| < \delta$ and $g \in G^{(r)}$. Then for $f \in C_0^r(\alpha, 1; a', b')$ we have

$$\begin{aligned}
\big|\Delta_h^2 f^{(r)}(x)\big| &\leqslant \big|\Delta_h^2\big(f^{(r)}(x) - g^{(r)}(x)\big)\big| + \big|\Delta_h^2 g^{(r)}(x)\big| \\
&\leqslant 2^2 \big\| f^{(r)} - g^{(r)}\big\|_{C[a',b']} + h^2 \big\| g^{(r+2)}\big\|_{C[a',b']} \\
&\leqslant 4C_9 K_r\big(h^2, f\big) \leqslant C_{10} h^\alpha,
\end{aligned}$$

which implies that $\omega_2(f^{(r)}, \delta; a, b) = \sup_{|h|<\delta} |\Delta_h^2 f^{(r)}(x)| \leqslant C_{10}\delta^\alpha$, i.e., $f^{(r)} \in \mathrm{Lip}^*(\alpha, a, b)$.

This completes the proof of the lemma. □

Theorem 4.3. *Let* $0 < \alpha < 2$ *and* $0 < a_1 < a_2 < b_2 < b_1 < \infty$. *If* $f \in C_\gamma[0, \infty)$, *then in the following statements, the implication* (i) $\Rightarrow$ (ii) *holds*:

(i) $\|B_n^{(r)}(f,*) - f^{(r)}\|_{C[a_1,b_1]} = O(n^{-\alpha/2})$;
(ii) $f^{(r)} \in \mathrm{Lip}^*(\alpha, a_2, b_2)$.

Proof. We shall prove the theorem in two cases.

Case I. When $0 < \alpha \leqslant 1$. Let a', a'', b', b'' be positive numbers such that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$.

Also let $g \in C_0^\infty$ with $\operatorname{supp} g \subset [a'', b'']$ such that $g(x) = 1$ on $[a_2, b_2]$. Then by linearity property, for $x \in [a', b']$ with $D \equiv \frac{d}{dx}$, we have

$$\begin{aligned}
B_n^{(r)}(fg, x) - (fg)^{(r)}(x) &= D^r\big(B_n\big([fg](t) - [fg](x), x\big)\big) \\
&= D^r\big(B_n\big(f(t)\big[g(t) - g(x)\big], x\big)\big) \\
&\quad + D^r\big(B_n\big(g(x)\big[f(t) - f(x)\big], x\big)\big) \\
&= A_1 + A_2, \quad \text{say.}
\end{aligned}$$

By Leibniz theorem, we have

$$\begin{aligned}
A_1 &= \frac{\partial^r}{\partial x^r} \int_0^\infty W_n(t,x) f(t)\big[g(t) - g(x)\big]\,dt \\
&= \sum_{i=0}^r \binom{r}{i} \int_0^\infty W_n^{(i)}(t,x) \frac{\partial^{r-i}}{\partial x^{r-i}} \big\{f(t)\big[g(t) - g(x)\big]\big\}\,dt
\end{aligned}$$

$$= -\sum_{i=0}^{r-1}\binom{r}{i}\int_0^\infty W_n^{(i)}(t,x)f(t)g^{(r-i)}(x)\,dt + \int_0^\infty W_n^{(r)}(t,x)f(t)\big[g(t)-g(x)\big]\,dt$$
$$= -\sum_{i=0}^{r-1}\binom{r}{i}g^{(r-i)}(x)B_n^{(i)}(f,x) + \int_0^\infty W_n^{(r)}(t,x)f(t)\big[g(t)-g(x)\big]\,dt$$
$$= A_3 + A_4, \quad \text{say.}$$

Using Theorem 3.3, we get

$$A_3 = -\sum_{i=0}^{r-1}\binom{r}{i}g^{(r-i)}(x)f^{(i)}(x) + O\big(n^{-\alpha/2}\big) \quad \text{uniformly in } x \in [a', b']$$
$$= -(fg)^{(r)}(x) + g(x)f^{(r)}(x) + O\big(n^{-\alpha/2}\big).$$

By Taylor's expansion of $f(t)$ and $g(t)$, we have

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + O(t-x)^r$$

and

$$g(t) = \sum_{i=0}^{r+1}\frac{g^{(i)}(x)}{i!}(t-x)^i + O(t-x)^{r+1}.$$

Substituting the above expansions in A_4 and using Theorem 3.2, Schwarz inequality and Lemma 2.2, we obtain

$$A_4 = \sum_{i=1}^{r}\frac{g^{(i)}(x)}{i!}\frac{f^{(r-i)}(x)}{(r-i)!}r! + O\big(n^{-1/2}\big)$$
$$= \sum_{i=1}^{r}\binom{r}{i}g^{(i)}(x)f^{(r-i)}(x) + O\big(n^{-\alpha/2}\big) \quad \text{uniformly in } x \in [a', b']$$
$$= (fg)^{(r)}(x) - g(x)f^{(r)}(x) + O\big(n^{-\alpha/2}\big).$$

Again by Leibniz theorem and using Theorem 3.3 and the hypothesis that (i) holds, we obtain

$$A_2 = \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_n^{(i)}(t,x)\frac{\partial^{r-i}}{\partial x^{r-i}}\big\{g(x)\big[f(t)-f(x)\big]\big\}\,dt$$
$$= \sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(x)B_n^{(i)}(f,x) - (fg)^{(r)}(x)$$
$$= \sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(x)f^{(i)}(x) - (fg)^{(r)}(x) + O\big(n^{-\alpha/2}\big)$$
$$= O\big(n^{-\alpha/2}\big) \quad \text{uniformly in } x \in [a', b'].$$

Finally collecting the estimates of A_3, A_4, A_2 and A_1, we get

$$\left\| B_n^{(r)}(fg, *) - (fg)^{(r)} \right\|_{C[a',b']} = O\left(n^{-\alpha/2}\right).$$

Thus, using Lemmas 4.1 and 4.2, we obtain

$$(fg)^{(r)} \in \mathrm{Lip}^*(\alpha, a', b'),$$

which implies that $f^{(r)} \in \mathrm{Lip}^*(\alpha, a_2, b_2)$ because $g(x) = 1$ on $[a_2, b_2]$. This completes the proof of the implication (i) $\Rightarrow$ (ii) for the case $0 < \alpha \leqslant 1$.

Case II. When $1 < \alpha < 2$. Let a^*, a^{**}, b^*, b^{**} be positive numbers satisfying $a_1 < a^* < a^{**} < a_2 < b_2 < b^{**} < b^* < b_1$. If $\delta > 0$ then $1 - \delta < 1$. Therefore by Case I, we have $f^{(r)} \in \mathrm{Lip}^*(1 - \delta, a^*, b^*)$. Let $g \in C_0^\infty$ with $\operatorname{supp} g \subset (a^{**}, b^{**})$ be such that $g(x) = 1$ on $[a_2, b_2]$. If $\psi(t)$ denotes the characteristic function of the interval $[a^*, b^*]$, then we have

$$\begin{aligned}
&\left\| B_n^{(r)}(fg, x) - (fg)^{(r)}(x) \right\|_{C[a^{**},b^{**}]} \\
&\quad \leqslant \left\| D^r \left\{ B_n\left(g(x)\left[f(t) - f(x)\right], x\right)\right\} \right\|_{C[a^{**},b^{**}]} \\
&\qquad + \left\| D^r \left\{ B_n\left(f(t)\left[g(t) - g(x)\right], x\right)\right\} \right\|_{C[a^{**},b^{**}]} \\
&\quad = T_1 + T_2, \quad \text{say}.
\end{aligned}$$

Using linearity property, Leibniz theorem, Theorem 3.3 and the hypothesis that (i) holds, we have

$$\begin{aligned}
T_1 &= \left\| D^r \left\{ g(x) B_n(f, x) - (fg)(x) B_n(1, x) \right\} \right\|_{C[a^{**},b^{**}]} \\
&= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(x) B_n^{(i)}(f, x) - (fg)^{(r)}(x) \right\|_{C[a^{**},b^{**}]} \\
&= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) - (fg)^{(r)}(x) \right\|_{C[a^{**},b^{**}]} + O\left(n^{-\alpha/2}\right) = O\left(n^{-\alpha/2}\right).
\end{aligned}$$

By Leibniz theorem and Theorem 3.2, we get

$$\begin{aligned}
T_2 &= \left\| -\sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) B_n^{(i)}(f, x) + B_n^{(r)}\left(f(t)\left[g(t) - g(x)\right]\psi(t), x\right) \right\|_{C[a^{**},b^{**}]} \\
&\quad + O\left(n^{-1}\right) \\
&= \| T_3 + T_4 \|_{C[a^{**},b^{**}]} + O\left(n^{-1}\right), \quad \text{say}.
\end{aligned}$$

Using Theorem 3.3, we obtain

$$\begin{aligned}
T_3 &= -\sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) + O\left(n^{-\alpha/2}\right) \quad \text{uniformly in } x \in [a^{**}, b^{**}] \\
&= -(fg)^{(r)}(x) + g(x) f^{(r)}(x) + O\left(n^{-\alpha/2}\right).
\end{aligned}$$

Using Taylor's expansion of $f(t)$, we get

$$\begin{aligned}
T_4 &= \int_0^\infty W_n^{(r)}(x,t)\big[f(t)\big[g(t)-g(x)\big]\psi(t)\big]\,dt \\
&= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(x,t)(t-x)^i\big[g(t)-g(x)\big]\psi(t)\,dt \\
&\quad + \int_0^\infty W_n^{(r)}(x,t)\left[\frac{f^{(r)}(\xi)-f^{(r)}(x)}{r!}\right](t-x)^r\big[g(t)-g(x)\big]\psi(t)\,dt \\
&\qquad \text{(where } \xi \text{ lying between } t \text{ and } x\text{)} \\
&= T_5 + T_6, \quad \text{say.}
\end{aligned}$$

Using Theorem 3.2, we obtain

$$\begin{aligned}
T_5 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(x,t)(t-x)^i\big[g(t)-g(x)\big]\,dt + O\big(n^{-1}\big) \\
&\qquad \text{uniformly in } x\in[a^{**},b^{**}] \\
&= T_7 + O\big(n^{-1}\big), \quad \text{say.}
\end{aligned}$$

Again using Taylor's expansion of $g\in C_0^\infty$, we have

$$\begin{aligned}
T_7 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(x,t)(t-x)^i \\
&\quad \times\left[g(x)+\sum_{j=1}^{r+2}\frac{g^{(j)}(x)}{j!}(t-x)^j+\varepsilon(t,x)(t-x)^{r+2}-g(x)\right]dt \\
&\qquad \text{(where } \varepsilon(t,x)\to 0 \text{ as } t\to x\text{)} \\
&= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=1}^{r+2}\frac{g^{(j)}(x)}{j!}\int_0^\infty W_n^{(r)}(x,t)(t-x)^{i+j}\,dt \\
&\quad + \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(x,t)\varepsilon(t,x)(t-x)^{i+r+2}\,dt \\
&= T_8 + T_9, \quad \text{say.}
\end{aligned}$$

Since $\int_0^\infty W_n^{(r)}(x,t)(t-x)^k\,dt = 0\ \forall k<r$. Therefore by Theorem 3.2 and Corollary 2.3, we obtain

$$\begin{aligned}
T_8 &= \sum_{j=1}^{r}\frac{g^{(j)}(x)}{j!}\frac{f^{(r-j)}(x)}{(r-j)!}r! + O\big(n^{-1}\big) \quad \text{uniformly in } x\in[a^{**},b^{**}] \\
&= \sum_{j=1}^{r}\binom{r}{j}g^{(j)}(x)f^{(r-j)}(x) + O\big(n^{-1}\big) \\
&= (gf)^{(r)}(x) - g(x)f^{(r)}(x) + O\big(n^{-1}\big).
\end{aligned}$$

Also as in the proof of Theorem 3.1, it can be easily shown that

$$T_9 = O\left(n^{-\alpha/2}\right) \quad \text{uniformly in } x \in [a^{**}, b^{**}].$$

Next using Lemma 2.5, the mean value theorem, Schwarz inequality and Lemma 2.2, we have

$$\begin{aligned}
&\|T_6\|_{C[a^{**},b^{**}]} \\
&\quad \leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+2)^{i+j} \left\| \frac{Q_{i,j,r}(x)}{[x(1+x)]^r} \int_0^\infty W_n(x,t)|t-x|^{\delta+r+1} \right. \\
&\qquad \left. \times \frac{|f^{(r)}(\xi) - f^{(r)}(x)|}{r!} \left|g'(\varsigma)\right| \psi(t)\,dt \right\|_{C[a^{**},b^{**}]} \\
&\qquad\quad (\text{where } \varsigma \text{ lying between } t \text{ and } x) \\
&\quad = O\left(n^{-\delta/2}\right), \quad \text{where } \delta \text{ is chosen in such a way that } 0 \leqslant \delta \leqslant 2-\alpha.
\end{aligned}$$

Finally, collecting the estimates of T_1 to T_9, we get

$$\left\| B_n^{(r)}(fg,x) - (fg)^{(r)}(x) \right\|_{C[a^{**},b^{**}]} = O\left(n^{-\alpha/2}\right).$$

Since supp $fg \subset (a^{**}, b^{**})$, therefore by Lemmas 4.1 and 4.2, we get $(fg)^{(r)} \in \text{Lip}^*(\alpha, a^{**}, b^{**})$, which implies that $f^{(r)} \in \text{Lip}^*(\alpha, a_2, b_2)$ because $g(x) = 1$ on $[a_2, b_2]$. This completes the proof of the implication (i) $\Rightarrow$ (ii) for the case $1 < \alpha < 2$.

Hence the proof of the theorem is finished. □

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